Some congruences concerning second order linear recurrences

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Abstract. Let U_n and V_n (n=0,1,2,...) be sequences of integers satisfying a second order linear recurrence relation with initial terms U_0=0, U_1=1, V_0=2, V_1=A. In this paper we investigate the congruence properties of the terms U_{nk} and V_{nk}, where the moduli are powers of U_n and V_n.

Let U_n and V_n $(n = 0, 1, 2, \ldots)$ be second order linear recursive sequences of integers defined by

$$U_n = AU_{n-1} - BU_{n-2} \quad (n > 1)$$

and

$$V_n = AV_{n-1} - BV_{n-2} \quad (n > 1),$$

where A and B are nonzero rational integers and the initial terms are $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = A$. Denote by α, β the roots of the characteristic equation $x^2 - Ax + B = 0$ and suppose $D = A^2 - 4B \neq 0$ and hence that $\alpha \neq \beta$. In this case, as it is well known, the terms of the sequences can be expressed as

$$(1) \qquad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

for any $n \geq 0$.

Many identities and congruence properties are known for the sequences U_n and V_n (see, e.g. [1], [4], [5] and [6]). Some congruence properties are also known when the modulus is a power of a term of the sequences (see [2], [3], [7] and [8]). In [3] we derived some congruences where the moduli was U_n^3, V_n^2 or V_n^3. Among other congruences we proved that

$$U_{nk} \equiv kB^{n\frac{k-1}{2}} U_n \pmod{U_n^3}$$

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when k is odd and a similar congruence for even k. In this paper we extend the results of [3]. We derive congruences in which the moduli are product of higher powers of U_n and V_n.

Theorem. *Let U_n and V_n be second order linear recurrences defined above and let $D = A^2 - 4B$ be the discriminant of the characteristic equation. Then for positive integers n and k we have*

1. $U_{nk} \equiv kB^{\frac{k-1}{2}n}U_n + \frac{k(k^2-1)}{24}DB^{\frac{k-3}{2}n}U_n^3 \pmod{D^2U_n^5}$, k odd,
2. $U_{nk} \equiv \frac{k}{2}B^{\frac{k-2}{2}n}V_nU_n + \frac{k(k^2-4)}{48}DB^{\frac{k-4}{2}n}V_nU_n^3 \pmod{D^2V_nU_n^5}$, k even,
3. $V_{nk} \equiv k(-1)^{\frac{k-1}{2}}B^{\frac{k-1}{2}n}V_n + \frac{k(k^2-1)}{24}(-1)^{\frac{k-3}{2}}B^{\frac{k-3}{2}n}V_n^3 \pmod{V_n^5}$, k odd,
4. $V_{nk} \equiv 2(-1)^{\frac{k}{2}}B^{\frac{k}{2}n} + \frac{k^2}{4}(-1)^{\frac{k-2}{2}}B^{\frac{k-2}{2}n}V_n^2 \pmod{V_n^4}$, k even,
5. $U_{nk} \equiv U_n(-1)^{\frac{k-1}{2}}B^{\frac{k-1}{2}n} + \frac{k^2-1}{8}(-1)^{\frac{k-3}{2}}B^{\frac{k-3}{2}n}U_nV_n^2 \pmod{U_nV_n^4}$, k odd,
6. $U_{nk} \equiv \frac{k}{2}(-1)^{\frac{k-2}{2}}B^{\frac{k-2}{2}n}U_nV_n + \frac{k(k^2-4)}{48}(-1)^{\frac{k-4}{2}}B^{\frac{k-4}{2}n}U_nV_n^3 \pmod{U_nV_n^5}$, k even,
7. $V_{nk} \equiv B^{\frac{k-1}{2}n}V_n + \frac{k^2-1}{8}DB^{\frac{k-3}{2}n}V_nU_n^2 \pmod{D^2V_nU_n^4}$, k odd,
8. $V_{nk} \equiv 2B^{\frac{k}{2}n} + \frac{k^2}{4}B^{\frac{k-2}{2}n}DU_n^2 \pmod{D^2U_n^4}$, k even.

We note that the congruences of [3] follow as consequences of this theorem.

For the proof of the Theorem we need some auxiliary results which are known (see e.g. [6]) but we show short proofs for them. In the followings we suppose that $A > 0$ and hence that

$$\alpha = \frac{A + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{D}}{2},$$

so that $\alpha - \beta = \sqrt{D}$, $\alpha + \beta = A$, $\alpha\beta = B$ and hence by (1)

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{D}} \tag{2}$$

Lemma 1. *For any integer $n \geq 0$ we have*

$$U_{3n} = 3U_nB^n + DU_n^3.$$

Proof. By (2), using that $\alpha\beta = B$, we have to prove that

$$\frac{\alpha^{3n} - \beta^{3n}}{\sqrt{D}} = 3 \cdot \frac{\alpha^n - \beta^n}{\sqrt{D}}(\alpha\beta)^n + D\left(\frac{\alpha^n - \beta^n}{\sqrt{D}}\right)^3,$$

which follows from $\alpha^{3n} - \beta^{3n} = 3(\alpha^n - \beta^n)\alpha^n\beta^n + (\alpha^n - \beta^n)^3$.

Lemma 2. *For any non-negative integers m and n we have*

$$U_{m+2n} = V_n U_{m+n} - B^n U_m.$$

Proof. Similarly as in the proof of Lemma 1,

$$\frac{\alpha^{m+2n} - \beta^{m+2n}}{\sqrt{D}} = (\alpha^n + \beta^n)\frac{\alpha^{m+n} - \beta^{m+n}}{\sqrt{D}} - (\alpha\beta)^n \frac{\alpha^m - \beta^m}{\sqrt{D}}$$

is an identity which by (1) and (2), implies the lemma.

Lemma 3. *For any $n \geq 0$ we have*

$$V_{2n} = 2B^n + DU_n^2 = V_n^2 - 2B^n \quad \text{and} \quad U_{2n} = U_n V_n.$$

Proof. The identities

$$\alpha^{2n} + \beta^{2n} = 2(\alpha\beta)^n + D\left(\frac{\alpha^n - \beta^n}{\sqrt{D}}\right)^2 \quad \text{and} \quad \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{D}} = \frac{\alpha^n - \beta^n}{\sqrt{D}}(\alpha^n + \beta^n)$$

prove the lemma.

Proof of the Theorem. We prove the first congruence of the Theorem by double induction on k. For $k = 1$ and $k = 3$, by Lemma 1, the congruence is an identity. Suppose the congruence holds for k and $k + 2$, where $k \geq 1$ is odd. Then by Lemma 2 and 3 we have

$$\begin{aligned} U_{n(k+4)} &= U_{nk+4n} = V_{2n}U_{nk+2n} - B^{2n}U_{nk} \\ &= (2B^n + DU_n^2)U_{n(k+2)} - B^{2n}U_{nk} \\ &\equiv (2B^n + DU_n^2)Q - B^{2n}R \pmod{D^2U_n^5}, \end{aligned} \tag{3}$$

where

$$Q = (k+2)B^{\frac{k+1}{2}n}U_n + \frac{(k+2)((k+2)^2 - 1)}{24}DB^{\frac{k-1}{2}n}U_n^3 \tag{4}$$

and

$$R = kB^{\frac{k-1}{2}n}U_n + \frac{k(k^2-1)}{24}DB^{\frac{k-3}{2}n}U_n^3. \tag{5}$$

After some calculation (3), (4) and (5) imply

$$U_{n(k+4)} \equiv U_n T + U_n^3 S \pmod{D^2 U_n^5}, \tag{6}$$

where

$$T = (2(k+2)-k)\, B^{\frac{k+3}{2}n} = (k+4)B^{\frac{(k+4)-1}{2}}$$

and

$$S = (k+2)DB^{\frac{k+1}{2}n} + 2\frac{(k+2)\left((k+2)^2-1\right)}{24}DB^{\frac{k+1}{2}n}$$
$$-\frac{k(k^2-1)}{24}DB^{\frac{k+1}{2}n} = \frac{(k+4)\left((k+4)^2-1\right)}{24}DB^{\frac{(k+4)-3}{2}n},$$

and so by (6),

$$U_{n(k+4)} \equiv (k+4)B^{\frac{(k+4)-1}{2}}U_n$$
$$+\frac{(k+4)\left((k+4)^2-1\right)}{24}DB^{\frac{(k+4)-3}{2}n}U_n^3 \pmod{D^2U_n^5}.$$

Hence the congruence holds also for $k+4$ and for any odd positive integer k.

The other congruences in the Theorem can be proved similarly using Lemma 1, 2, 3 and the identities

$$\begin{aligned}
U_{2n} &= V_n U_n, \\
V_{2n} &= V_n^2 - 2B^n = 2B^n + DU_n^2, \\
U_{3n} &= U_n V_n^2 - B^n U_n, \\
V_{3n} &= V_n^3 - 3B^n V_n = B^n V_n + DV_n U_n^2, \\
U_{4n} &= U_n V_n^3 - 2B^n U_n V_n, \\
V_{4n} &= V_n^4 - 4B^n V_n^2 + 2B^{2n}.
\end{aligned}$$

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